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BRITTLE CLEAVAGE OF A PIECEWISE-HOMOGENEOUS ELASTIC MEDIUM*

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Stationary pre-Rayleigh motion of a rigid body along a straight line connecting two elastic half-planes with the formation of a crack and a cavern is investigated. The contact between the edges in a small zone of the edge of the crack and outside the cavern at a large distance from the wedge is taken into account by the method of joining asymptotic expansions. As is shown, the ratios between the characteristic lengths are, respectively, quite small and quite large parameters if the wedge velocity is not close to the Rayleigh velocity, which specifies the advisability of using such an approach.

1. An absolutely blunt rigid wedge of thickness $h(x)$, $|x| \leqslant 1$ moves without friction at a constant velocity c along the interface $y=0$, $|x| < \infty$ of two elastic media occupying the half-plane $y > 0$ (medium 1) and $y < 0$ (medium 2) (Fig.1). A crack of length $a-1$ is formed ahead of the wedge and a cavity for $-\infty < x < -1$. The crack edges and the cavity do not interact and are force-free (an a priori assumption). The sides of the wedge are completely adjacent to to medium. Total contact conditions are satisfied for $x > a$, $y=0$.

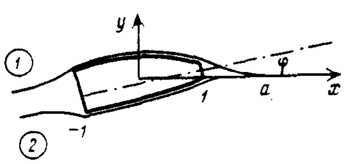

Fig.1

It is required to determine the steady stress field $\sigma_{km}{}^j(x, y)$ and the displacement field $U_m{}^j(x, y)$ from the following boundary conditions $(y=0)$:

$$
\begin{gathered}
U_{2,x}^j = h_j'(x) - \varphi,\ \sigma_{12}{}^j = 0,\ \sigma_{22}{}^j \leqslant 0,\ |x| < 1 \\
\sigma_{k2}{}^j = 0,\ [U_2] \geqslant 0,\ 1 < x < a \cup x < -1,\ [\sigma_{k2}] = [U_k] = 0, \\
x > a \\
[U_2(1)] = h(1), \quad \int_{-1}^{1} [\sigma_{22}] \left\| {}^1_x \right\| dx = 0 \quad (k, m, j = 1, 2)
\end{gathered} \tag{1.1}
$$

Here $h_j = h_j(x)$ is the equation of the wedge surface relative to some of its axes, $h_j(x)$ are Hölder-continuous functions, $h = h_1 - h_2$, $h(1) \ll a-1$, $|h_j'(x)| \ll 1$, $|x| < 1$, φ is the angle of rotation of the wedge axis, the subscript j defines the mediums, square brackets denote the jump in a quantity on passing from medium 1 into medium 2, the prime denotes ordinary differentiation, and the coordinate system is moving.

It is convenient to express the stresses and the derivatives of the displacements in dynamic linear elasticity theory (the plane problem, steady subsonic mode) in terms of analytic functions $\chi_m{}^j(z_{kj})$ of the complex variable $z_{kj} = x + i\beta_{kj}y$ by means of formulas /1/ (representations close to the representations in /2/). On the interface $z_{mj} = x$

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$$U_{1,x}^j = -\operatorname{Re}(b_{2j}\chi_1{}^j + a_j\chi_2{}^j),\quad U_{2,x}^j = \operatorname{Im}(a_j\chi_1{}^j + b_{1j}\chi_2{}^j) \tag{1.2}$$
$$\sigma_{12}{}^j = \operatorname{Im}\chi_1{}^j,\quad \sigma_{22}{}^j = \operatorname{Re}\chi_2{}^j,\quad \beta_{mj} = \sqrt{1 - c^2/c_{mj}^2},\quad 2\beta_j = 1 + \beta_{2j}^2$$
$$2\mu_j R_j(a_j, b_{mj}) = (\beta_{1j}\beta_{2j} - \beta_j, \beta_{mj}(1 - \beta_j)),\quad R_j = \beta_{1j}\beta_{2j} - \beta_j^2$$

where μ_j is the shear modulus, c_{1j} and c_{2j} are the expansion and shear wave velocitites, $R_j(c)$ is a Rayleigh functions (c_{Rj}, the unique positive roots of the Rayleigh equation $R_j(c) = 0$, are the velocities of the natural surface waves $c_R = \min(c_{R1}, c_{R2})$).

We will seek the solution in the energy class of functions with finite displacements everywhere. Then the following estimates hold for the behaviour of the functions $\chi_m{}^j(z)$ at singularities ($z = x + iy$ is an auxiliary variable):

$$|\chi_m{}^j| < \frac{\text{const}}{|z - z_k|^{1/2}},\quad z \to z_k,\quad |\chi_m{}^j| < \frac{\text{const}}{|z|^{1+\varepsilon}},\quad z \to \infty \tag{1.3}$$
$$z_1 = -1,\ z_2 = 1,\ z_3 = a,\ \varepsilon > 0\ (k = 1, 2, 3;\ m, j = 1, 2).$$

We first examine an auxiliary problem for the functions $\chi^j(z)$ that removes the inhomogeneity in the first of conditions (1.1)

$$b_{1j} \operatorname{Im} \chi^j(x) = h_j'(x) + \varphi,\ |x| < 1,\ \operatorname{Re}\chi^j(x) = 0,\ |x| > 1.$$

Taking account of (1.3), the unqiue solution of this particular Keldysh-Sedov problem takes the form /3/

$$b_{1j}\chi^j = i\varphi\left(1 - \frac{z}{\sqrt{z^2 - 1}}\right) + \frac{i}{\pi\sqrt{z^2 - 1}}\int_{-1}^{1}\frac{h_j'(t)\sqrt{1 - t^2}\,dt}{t - z} \tag{1.4}$$
$$\lim_{z\to\infty} z^{-1}\sqrt{z^2 - 1} = 1.$$

After removing the state of stress given by a solution of the form $\chi_1{}^j = 0,\ \chi_2{}^j = \chi^j$, the condition $[\operatorname{Im}\chi_1] = 0$ for $|x| < \infty$ and the conditions $\operatorname{Im}\chi_2{}^j = 0, [\operatorname{Re}\chi_2] = 0$ in the intervals supplementing each other to almost the full real axis (we do not change the function notation) enable the following conclusion to be drawn when (1.3) is taken into account /4/:

$$\chi_1{}^1(z) = -\overline{\chi_1{}^2(\bar{z})} \equiv \chi_1(z),\quad \chi_2{}^1(z) = \overline{\chi_2{}^2(\bar{z})} \equiv \chi_2(z),\quad y > 0. \tag{1.5}$$

The bar denotes the complex conjugate.

The conditions $[\operatorname{Im}\chi_1] = 0, \ldots$ follow directly from (1.5) if, in addition, the function $\chi_2{}^2(z)$ is the analytic continuation of the function $\chi_2{}^1(z)$ through the segment $|x| \leqslant 1, y = 0$. The converse can be shown by first writing the solutions from the class (1.3) for the following auxiliary boundary value problems for the functions $\chi_1{}^j(z)$ ($y = 0$):

$$\operatorname{Im}\chi_1{}^j = r(x),\ |x| < \infty$$

and the functions $\chi_2{}^j(z)$

$$\operatorname{Im}\chi_2{}^j = 0,\ |x| < 1,\ \operatorname{Re}\chi_2{}^j = s(x),\ |x| > 1$$

where $r(x)$ and $s(x)$ are certain real functions satisfying the Hölder condition.

Relationships (1.5) reduce the number of unknown functions to two. The fundamental problem, the Riemann-Hilbert problem /5/, is obtained from (1.1),(1.2),(1.4),(1.5): it is required to find a vector function $\chi = (\chi_1, \chi_2)$ from the class (1.3) that is holomorphic in the upper half-plane of z and satisfies the following conditions on the boundary $y = 0$:

$$\operatorname{Im}(H\chi) = (f(x), 0),\ x > a,\ \operatorname{Im}\chi_1 = \operatorname{Im}\chi_2 = 0,\ |x| < 1 \tag{1.6}$$
$$\operatorname{Im}\chi_1 = \operatorname{Re}\chi_2 = 0,\ (x < -1) \cup (1 < x < a)$$
$$H = \begin{Vmatrix} d & p \\ iq & id \end{Vmatrix},\quad f(x) = \frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1 - t^2}{x^2 - 1}}\,\frac{h'(t)\,dt}{x - t}$$
$$d = a_1 - a_2 < 0,\ p = b_{11} + b_{12},\ q = b_{21} + b_{22}.$$

Additional conditions in the form of inequalities from (1.1) are to be confirmed by superposition of the solutions (1.4), and (1.6). Problem (1.3),(1.6) contains three kinds of boundary conditions and four singularities. The general method of constructing a closed solution of the related Riemann-Hilbert boundary value problem for vector functions with a greater number of kinds of boundary conditions than two is not known. A method of solution is proposed in /4/ (in Cauchy type integrals) that extends to the case when the boundary conditions of the problem reduce to the form (1.6), where the first and third conditions of (1.6) can alternate on an arbitrary system of segments, "diluted" by a segment on which the second condition in (1.6) is satisfied, by any method.

Following /4/ we carry out such a sequence of actions. We continue the vector $\chi(z)$ analytically through the segment $|x| \leqslant 1$. We map the plane z with the slits $|x| \geqslant 1$, $y = 0$ conformally in the upper half-plane $\omega = \xi + i\eta = z + \sqrt{z^2 - 1}$ $(2z = \omega + \omega^{-1})$: the half-plane $\operatorname{Im} z > 0$ goes over into the exterior, and the half-plane $\operatorname{Im} z < 0$ into the interior of the semicircles $|\omega| = 1$, $\operatorname{Im} \omega > 0$ (Fig.2). We continue the vector $\chi(\omega)$ (without renotation) through the real axis according to the rule $\chi_k(\omega) = (-1)^{k+1}\overline{\chi_k(\bar{\omega})}$ and we diagonalize the coefficient matrix of the conjugate being obtained for this problem. As a result of the linear substitution

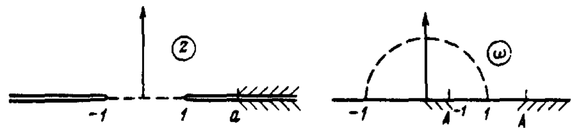

Fig.2

$$\chi_1 = W_1 + W_2,\ \chi_2 = s(W_1 - W_2)\ (s = \sqrt{q/p}) \tag{1.7}$$

we arrive at such a conjugate problem for the vector function $\mathbf{W} = (W_1, W_2)$ (we indicate narrowing on the axis $x = 0$ from above (below) by the superscript plus (minus)):

$$\begin{gathered}\mathbf{W}^+ = \Lambda\mathbf{W}^- + 2i\mathbf{W}^\circ(\xi),\quad \xi \in L_1,\quad \mathbf{W}^+ = \mathbf{W}^-,\quad \xi \in L_2 \\ \Lambda = \begin{Vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{Vmatrix},\quad \mathbf{W}^\circ = \frac{1}{2}\begin{Vmatrix}(d + \sqrt{pq})^{-1}\\ (d - \sqrt{pq})^{-1}\end{Vmatrix} g(\xi),\quad \lambda = \frac{\sqrt{pq} - d}{\sqrt{pq} + d} > 1 \\ \lambda_1 = \lambda_2^{-1} = -\lambda,\ g = f[x(\xi)],\ \xi > A,\ g(\xi) = -g(1/\xi),\ 0 < \\ \xi < A^{-1} \\ A = a + \sqrt{a^2 - 1},\ L_1 = \{\xi: 0 < \xi < A^{-1} \cup \xi > A\} \\ L_2 = \{\xi: \xi < 0 \cup A^{-1} < \xi < A\}.\end{gathered} \tag{1.8}$$

Additional conditions of the problem include the continuation conditions

$$W_k(\omega) = \overline{W_k(1/\bar{\omega})},\ W_k(\omega) = \overline{W_m(\bar{\omega})}\ (k, m = 1, 2;\ m \neq k) \tag{1.9}$$

and estimates resulting from (1.3).

The generalized conjugate problem (1.7)-(1.9) is equivalent to problem (1.3),(1.6), all the matrices are not degenerate for $0 \leqslant c < c_R$. The sole singularity of the problem (1.8), (1.9) from the ordinary Hilbert problem /5/ obtained from the Riemann-Hilbert problem, is the presence of still another additional condition in (1.9), namely, the condition of inversion of $W_k(\omega)$. It replaces the second boundary condition in (1.6). Therefore, the meaning of the above-mentioned transformations is elimination of one kind of boundary conditions out of the number of fundamental conditions and transferring it into an additional condition. It is essential that the coefficients of the problem for the function $\chi(\omega)$ in the system of segments for $|\xi| > 1$ be equal, respectively, to the coefficients on the system of segments symmetric with respect to the circle $|\omega| = 1$ for $|\xi| < 1$; this is ensured by the particular form of the matrix $H = \{h_{km}\}$

$$h_{1m} = \operatorname{Re} h_{1m},\ h_{2m} = i \operatorname{Im} h_{2m}\ (k, m = 1, 2).$$

The functions $W_k(\omega)$ here have simple poles at the points $\omega = \pm 1$ while $\chi_k(z)$ for $z = \pm 1$ would have exponents of number $(0, -1/2)$, i.e., uneliminable singularities at these points are transformed into eliminable ones. Consequently, the problem was split (the matrix Λ is diagonal). For each of the components of the vector W the solution is constructed by a method somewhat different from the traditional one /6/: we select the auxiliary functions such that the condition (1.9) could be realized in the most simple form, although the idea (factorization) would remain as before. The principle of a certain freedom of choice of the particular solution of the inhomogeneous problem is used here. The constraints (1.9) superpose additional relations on the free coefficients of the solution of the homogeneous problem corresponding to (1.8). There remains consequently one independent real constant, which is in agreement with the general theorem /5/ about the number of these constants in the initial problem (1.3),(1.7).

The general solution of (1.3),(1.7)-(1.9) is given by the formulas

$$W_k(\omega) = \Pi_k(\omega)\{F(\omega) I_k(\omega) - i(-1)^k C_0 G(\omega)\} \tag{1.10}$$

$$\Pi_k = \frac{[(A-\omega)/(A-\omega^{-1})]^{i\alpha_k}}{(A+A^{-1}-\omega-\omega^{-1})^{1/2}}, \quad F = \frac{\omega}{\omega+1} = (\omega-1)G, \quad \alpha_k = \frac{\ln|\lambda_k|}{2\pi}$$

$$-\pi < \arg(A-\omega^{\pm 1}) \leqslant \pi, \quad I_k = \frac{1}{\pi}\int_{L_1} \frac{W_k{}^\circ(\xi)\, d\xi}{F(\xi)\,\Pi_k{}^+(\xi)\,(\xi-\omega)} \ .$$

The auxiliary functions $\Pi_k(\omega)$ serve the purpose of factorization

$$\Pi_k{}^+(\xi)/\Pi_k{}^-(\xi) = \lambda_k, \ \xi \in L_1, \ \Pi_k{}^+(\xi)/\Pi_k{}^-(\xi) = 1, \ \xi \in L_2$$

and, moreover, possess the properties (we here present the properties of the functions $W_k{}^\circ(\xi)$)

$$\begin{gathered} \Pi_k(\omega) = \Pi_k(\overline{1/\overline{\omega}}), \ \Pi_k(\omega) = \overline{\Pi_m(\overline{\omega})} \\ W_k{}^\circ(\xi) = -W_k{}^\circ(1/\xi), \ \lambda_m W_k{}^\circ(\xi) = W_m{}^\circ(\xi) \ (k, m = 1, 2; \\ m \neq k) \end{gathered} \tag{1.11}$$

The auxiliary functions F and G ensure the presence of poles in the functions $W_k(\omega)$ at the points $\omega = -1$ and the existence of integrals $I_k(\omega)$ as well as, in combination with the functions $\Pi_k(\omega)$, compliance with conditions (1.9) and the estimates (1.3). For the realization of (1.9) they should be subjected to the following functional equation and the condition

$$\frac{F(\omega)}{\omega \overline{F(1/\overline{\omega})}} = \frac{F(\xi)}{\xi \overline{F(1/\xi)}} = \frac{\xi F(1/\xi)}{\overline{F(\xi)}}, \quad G(\omega) = -\overline{G(1/\overline{\omega})} \ .$$

The real constant C_0 and the angle φ are to be determined.

2. The constant φ is determined from the condition that the moment of the forces applied from the media to the wedge equals zero. The condition that the principal stress vector equals zero is satisfied automatically because of the conditions taken at infinity (as can be shown in the same way as in /7,8/ when there is no term $\sim 1/z$ in the asymptotic as $z \to \infty$ the principal stress vector applied to the boundary from outside turns out to be zero). The jump in the contact pressures $\sigma(x) = -[\sigma_{22}(x, 0)]$, $|x| < 1$ is determined just by the auxiliary solution

$$\sigma(x) = \varphi \frac{b_{11}^{-1} + b_{12}^{-1}}{\sqrt{1-x^2}}\, x - \frac{1}{\pi}\int_{-1}^{1} \frac{[b_{11}^{-1} h_1'(t) + b_{12}^{-1} h_2'(t)]\sqrt{1-t^2}\, dt}{\sqrt{1-x^2}\,(x-t)} \ .$$

From the condition $\int_{-1}^{1} x\sigma(x)\, dx = 0$ we obtain

$$\varphi = \frac{2}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{[b_{11}^{-1} h_1'(t) - b_{12}^{-1} h_2'(t)]\, x\sqrt{1-t^2}\, dt\, dx}{(b_{11}^{-1} + b_{12}^{-1})(x-t)\sqrt{1-x^2}} \ .$$

All the integrals that do not exist according to Riemann are understood in the principal value sense. If the media are identical $(b_{11} = b_{12})$ and the wedge is symmetric $(h_1' \equiv -h_2')$, then $\varphi = 0$.

We determine the constant C_0 by determining the jump in the displacement U_2 is some section of the wedge from the condition $[U_2(1)] = h(1)$, say. This is necessary since the problem is posed in derivatives of the displacements. To do this we integrate the complete value $[U_{2,x}] = -f(x) + p \operatorname{Im} \chi_2$, $1 < x < a$, where the first and second components are contributions of the solutions of the auxiliary and fundamental problems. On utilizing (1.11), we obtain

$$C_0 = [I_0 - h(1)]/I, \quad I_0 = \int_1^a \left\{f(x) - \sqrt{\frac{2pq}{a-x}}\, F[\xi(x)]\, E(x)\right\} dx \tag{2.1}$$

$$E = \operatorname{Im}\{I_1 e^{i\alpha\psi}\} = \frac{1}{\pi}\int_A^\infty \frac{W_1{}^\circ(t)}{F(t)}\left(\frac{t + t^{-1} - 2a}{\lambda}\right)^{1/2} \times$$

$$\frac{[\xi(x) - 1](t-1)\, dt}{[t - \xi(x)][1 - t\xi(x)]} + O(\alpha^2)$$

$$\psi = \ln\left|\frac{A-\xi}{A-\xi^{-1}}\right|, \quad \xi = x - \sqrt{x^2-1}$$

$$I = \int_1^a \left(\frac{pq}{2(a-x)(x^2-1)}\right)^{1/2} \cos(\alpha\psi)\, dx = \int_1^a \left(\frac{pq}{2(a-x)(x^2-1)}\right)^{1/2} dx + O(\alpha^2) \ .$$

It is here taken into account that the quantity $\alpha = \alpha_1$ is comparable to unity just for values of the velocity c close to c_R, otherwise $\alpha \ll 1$ and the principal part in the expansion in this parameter can be separated out. The contribution of the integration with respect to the small segment near the apex of the crack, where the oscillating singularity is essential, is estimated by the quantity $O(\alpha^2)$.

It remains to verify the inequality in (1.1). The conditions that $\sigma_{22}^j \leqslant 0$ be continuous impose constraints on the wedge geometry: physically it is clear that these conditions are not satisfied for all $h_j(x)$. The condition of non-intersection of the crack edges and the cavity $[U_2] \geqslant 0$ is certainly spoiled in a small neighbourhood of the point a and far from the wedge (as $x \to -\infty$). This defect in the solution is corrected below.

3. We will first consider the example of a wedge of rectangular profile. The complete solution will consist of the solution of the fundamental problem, where just from its homogeneous part $(h_j'(x) \equiv 0, \varphi = 0)$

$$\frac{\chi(z)}{iC_0G(z)} = \left\| \begin{matrix} \Pi_1(z) - \Pi_2(z) \\ s(\Pi_1(z) + \Pi_2(z)) \end{matrix} \right\|, \quad C_0 = -\frac{h}{l} \tag{3.1}$$

$$\Pi_k G = \frac{[(z + \sqrt{z^2-1} - A)/(z - \sqrt{z^2-1} - A)]^{i\alpha_k}}{2\sqrt{2(a-z)(z^2-1)}} \underset{z\to x+i0,\,|x|<1}{\longrightarrow} \frac{\exp[\alpha_k \theta(x)]}{2i\sqrt{2(a-x)(x^2-1)}}$$

$$0 \leqslant \theta(x) = \operatorname{Arc\,tg} \frac{\sqrt{1-x^2}}{A-x} \leqslant \pi .$$

Starting from (3.1), we compute the contact pressures that are identical on the upper and lower edges of the wedge $(|x| < 1)$

$$\sigma_{22} = \operatorname{Re} \chi_2(x) = \frac{sC_0}{\sqrt{2(a-x)(1-x^2)}} + O(\alpha^2) \underset{x\to\pm1\mp0}{\sim} \frac{N(\pm1)}{\sqrt{1\mp x}} \tag{3.2}$$

$$N(\pm1) = \frac{C_0}{2}\sqrt{\frac{q}{p(a\mp1)}}, \quad \sigma_{22}(x) < 0 .$$

We obtain for the jump in the vertical velocity of the edges for $x < -1$ and $1 < x < a$:

$$-c[U_{2,x}] = -\frac{cC_0\sqrt{pq}\cos[\alpha\psi_0(x)]}{\sqrt{2(a-x)(x^2-1)}} \operatorname{sgn} x \underset{x\to\pm1\pm0}{\sim} \frac{M(\pm1)}{\sqrt{\pm x - 1}}$$

$$\psi_0(x) = \ln\left| \frac{x - A + \sqrt{x^2-1}\operatorname{sgn} x}{x - A - \sqrt{x^2-1}\operatorname{sgn} x} \right|, \quad M(\pm1) = \mp\frac{cC_0}{2}\left(\frac{pq}{a\mp1}\right)^{1/2} .$$

The energy flux w /9/ can be computed by means of the concentration coefficients N and M

$$w = \frac{\pi}{2} NM, \quad w(\pm1) = \mp\frac{cq\pi C_0^2}{8(a\mp1)} . \tag{3.3}$$

The flux $w(1)$ is negative, the energy goes from the point $x = 1$ into the medium; the flux $w(-1)$ is positive, the energy is expended at the point $x = -1$. The sum of these energy fluxes is absorbed in the crack tip (the flux at infinity is zero) and, moreover, defines the lower bound of the magnitude of the horizontal force Q that must be applied to the wedge to maintain a given stationary motion, from the energetic inequality

$$cQ > -w(1) - w(-1) \Rightarrow Q > \frac{qC_0^2}{4(a^2-1)} . \tag{3.4}$$

To ensure equality on the right side of the first inequality in (3.4) the power expended in irreversible processes around the wedge angles should be added, and we obtain the energy balance equation for the wedge.

The physical explanation for the appearance of energy fluxes of different sign at the wedge angles can be the following. If the wedge is considered with "smoothed" angles (the stresses are continuous at the separation points), then the stresses normal to the wedge surface will evidently perform work of different sign above the medium near the forward and rear points of separation, while the wedge will experience frontal resistance.

In the general case, an integral of the projections of the normal stresses to the contour over the wedge contour on the x-axis should be added to the expression for the frontal resistance. The quantity Q is proportional to the square of the deformation, i.e., is referred to the place of the quantities neglected in formulating the linear problem of elasticity theory (to remove the boundary conditions on the non-deformable surface) and is determined a posteriori. For this reason the assertion about the principal vector of the external forces applied to the boundary being equal to zero remains valid.

The resistance to friction (the coefficients of friction are small) can be estimated by using the solution obtained as the zeroth approximation.

The stresses on the continuation of the crack $(x > a,\ y = 0)$ equal

$$\sigma_{22} + i\sigma_{12} = -\frac{C_0(\lambda^{1/2} + \lambda^{-1/2})}{2\sqrt{2(x-a)(x^2-1)}}\{s\cos[\alpha\ln\psi] + i\sin[\alpha\ln\psi]\} \sim$$
$$\frac{-K_2}{\sqrt{2\pi(x-a)}}\{\ldots\},\quad K_2 = C_0\left(\frac{\pi pq}{(pq-d^2)(a^2-1)}\right)^{1/2} < 0\,.$$

Formulas for the velocities as $x \to a - 0$ have an analogous structure (and of a non-planar wedge in the general case). If the velocity c is not too close to c_R, the domains where the condition $[U_2] \geqslant 0$ is violated are located in the zone of the crack tip, and in the domain of the cavity far from the wedge. Consequently, the solution obtained can be considered as an external expansion with respect to the neighbourhoods of the points $z = a, \infty$. We construct inner expansions below (the principal parts of the expansions are understood everywhere) by relying on the results in /10/. Sections of edge contact with slip in the intervals $-\infty < x < -L$ and $a - l < x < a$ are introduced here and it is assumed that $L \gg a,\ l \ll a - 1$, i.e., L and l are large and small parameters (to be determined). These assumptions will subsequently be justified by calculations, but now provide a foundation for the asymptotic approach /11/.

4. Since knowledge of one coefficient K_2 determines the principal part of the field locally, given by the external expansion, we use an analogy with /10/ for the solution of the inner problem having the domain of definition $|z - a| \ll a - 1$ and the overlap domain $l \ll |z - a| \ll a - 1$. We write the final result for the general case (Z is the inner variable)

$$\chi = -\frac{i|C_1|}{\sqrt{z-a}}\left\|\begin{matrix}\lambda\Omega^{i\alpha} + \Omega^{-i\alpha}\\ s(\lambda\Omega^{i\alpha} - \Omega^{-i\alpha})\end{matrix}\right\|$$
$$\sqrt{2\lambda}C_1 = C_0G(A) - iF(A)I_1(A) \equiv J_1 + iJ_2\ (\mathrm{Re}\,J_k = J_k,$$
$$\sqrt{1} = 1)$$
$$\Omega = 2Z + 1 - 2\sqrt{Z^2 - Z},\ Z = (z-a)/l\,.$$

For $|x| \ll l,\ y = 0$ we have

$$\sigma_{12} \sim K_2[2\pi(x-a)]^{-1/2},\ \sigma_{22} = O(1),\ x \to a + 0 \tag{4.1}$$
$$\sigma_{12} = 0,\quad \sigma_{22} \sim -\frac{d}{p}\frac{K_2}{[2\pi(a-x)]^{1/2}} < 0,\quad x \to a - 0$$
$$w(a) = \frac{c(pq-d^2)}{4p}K_2^2 = \frac{\pi cqC_0^2}{4(a^2-1)}\ \text{(a plane wedge)} \tag{4.2}$$
$$l = \frac{a^2-1}{2(a+\sqrt{a^2-1})}\exp\left[-\frac{\pi+\gamma}{2\alpha}\right],\quad \gamma = 2\,\mathrm{arctg}\frac{J_2}{J_1}$$
$$K_2 = -\sqrt{2\pi}(\lambda+1)|C_1|\,.$$

The equation $w(1) + w(-1) + w(a) = 0$ can be confirmed as the energy balance equation for the media. The angular distribution of the functions at the apex of a transverse shear crack on the interface is analysed in /1/.

The asymptotic forms (1.10), governing the behaviour of the solution as $z \to \infty$, $\mathrm{Im}\,z > 0$ and $\mathrm{Im}\,z < 0$ are needed for the merger in the neighbourhood $z = \infty$

$$W_k \sim B_k\omega^{i\alpha_k - 1/2},\quad \omega \to \infty,\quad W_k \sim \bar{B}_k\omega^{i\alpha_k + 1/2},\quad \omega \to 0 \tag{4.3}$$
$$B_1 = -\lambda B_2,\ \lambda^{1/2}B_2 = iI_1(0) - C_0 \equiv C_0 + \Gamma_1 + i\Gamma_2\ (\Gamma_k = \mathrm{Re}\,\Gamma_k)\,.$$

We note that the auxiliary solution does not take part in the construction of solutions in the neighbourhoods because of triviality. We seek the inner expansion in the domain $|z| \gg a$, $|Z| \ll L/a$ ($Z = e^{2\pi i}L/z$ is the inner variable in this case) with the overlap domain $a \ll |z| \ll L$, $1 \ll |Z| \ll L/a$ in a form analogous to /10/ by turning attention to the behaviour for $|z| \gg L$

$$\chi = \frac{Z^{1/2}}{\Omega - 1}\left\|\begin{matrix}W_1^* + W_2^*\\ s(W_1^* - W_2^*)\end{matrix}\right\|_{1 \ll Z \ll L/a} \sim \left\|\begin{matrix}W_1 + W_2\\ s(W_1 - W_2)\end{matrix}\right\|. \tag{4.4}$$

To seek the function $W_k^*(\Omega)$ we obtain a homogeneous problem of the form (1.8) with merger conditions as $\Omega \to 0, \infty$. Asymptotic equalities can be established $\omega \sim 8L\Omega$ ($\omega \to \infty$, $\Omega \to 0$), $\omega \sim \Omega/(8L)$ ($\omega \to 0$, $\Omega \to \infty$). that are valid in the overlap domain, and then the asymptotic form $W_k^*(\Omega)$ can be calculated from (4.3) and (4.4) as $\Omega \to 0, \infty$ and the problem can be solved. We consequently obtain (L, like l also, is determined from the condition for suppressing the singularity at the point of contact /4,10/)

$$\chi = \frac{i|B_2|}{(2z)^{1/2}}\left\|\begin{matrix}\lambda\Omega^{i\alpha} + \Omega^{-i\alpha}\\ s(\lambda\Omega^{i\alpha} - \Omega^{-i\alpha})\end{matrix}\right\|$$
$$L = \frac{1}{8}\exp\frac{\pi+\delta}{2\alpha},\quad \delta = 2\,\mathrm{arctg}\frac{\Gamma_2}{\Gamma_1 + C_0}\,.$$

For a plane wedge $\gamma = \delta = 0$, otherwise $\gamma, \delta = O(\alpha)$, i.e., the parameters L and l can be estimated by setting $\gamma = \delta = 0$. It follows from the above that $\sigma_{22}(x, 0) < 0$ at the edge contact sections for $x < -L$ and $a - l < x < a$.

Therefore, the problem of the cleavage of an elastic bimaterial along the interface containing six singularities and nine dimensionless parameters (if the wedge shape is characterized by two quantities) is solved approximately by splitting into four separate problems. The influence of the parameters a, h is traced directly in the final formulas.

5. We will now analyse the limit situations (in the other parameters) that is not so obvious. As $c \to c_R$, Q, K_2, $w(\pm 1)$, $w(a) \to 0$ follow from (3.3)-(3.5),(4.2) if the crack length is fixed. If there is a lower bound $|K_2| > K_* > 0$ then $a \to 1$ as $c \to c_R$. This is in qualitative agreement with the results /12/ where the motion of a semi-infinite wedge in a homogeneous medium is studied. However, it is necessary to refer to these deductions with care because as $c \to c_R$ we have $\lambda, \alpha \to \infty$, $L, l \to O(1)$ and the solution is meaningless. For near-Rayleigh velocities it is necessary to examine the problem mainly taking contact between the crack edges and the slot into account. Below we present values of L as functions of the parameters c, ν_1 (ν_1 is Poisson's ratio, and medium 2 is rigid)

ν_1		0.1			0.3			0.45	
c/c_{21}	0	0.7	0.85	0	0.7	0.9	0.8	0.9	0.93
L	$4\cdot 10^5$	92	5.7	$2\cdot 10^6$	$2\cdot 10^3$	6.3	$1.4\cdot 10^4$	56	7.4 .

The quantities $2Al^{-1}/(a^2 - 1)$ will be an order of magnitude greater here.

It is seen that over almost the whole interval $(0, c_R)$, L is a very lare number (because of the smallness of α in the exponent), where $L \approx 3$ for values of c differing by $\chi \approx 2.5\%$, 1.6%, 0.94% of c_R for $\nu_1 = 0.1$, 0.3, 0.45 (c_R/c_{21} 0.893, 0.927, 0.949), respectively. Hence, the approximate solution found has the power of the exact solution for $0 < c < c_R - \varepsilon$, $\varepsilon \approx 0.05 \cdot c_R$; further refinement has no practical meaning.

The problem of the motion of a non-symmetric wedge of finite length in a homogeneous medium with crack formation has apparently not been considered earlier. The passage to the limit $\mu_2 \to \mu_1$, $c_{m2} \to c_{m1}$, $m = 1, 2$ ($\lambda \to 1$, d, $\alpha \to 0$, $p \to 2b_{11}$, $q \to 2b_{21}$, $L \to \infty$, $l \to 0$) is of interest. The domains of definition of the inner expansions in the external coordinates here degenerate into a point, the external expansion becomes the exact solution, oscillations drop out, and (1.10), (3.1) simplify because

$$\Pi_k = [2(a - z)]^{-1/2}, \quad W_k^c = {}^1/_2 (-1)^{k-1} g(\xi)/\sqrt{pq}\,.$$

The approximate formulas (2.1) and (3.2) revert to exact formulas and the expressions for the stress components on the continuation of the crack take the form

$$\sigma_{22} = \frac{\sqrt{2}\, s\{F[\xi(x)] J_1[\xi(x)] - C_0 G[\xi(x)]\}}{\sqrt{x - a}}, \quad \sigma_{12} = \frac{f(x)}{\sqrt{pq}}$$

formulas (4.1) become meaningless while (3.3),(3.4) and (4.2) are conserved.

For $c = c_d$, where c_d is the root of $d(c) = 0$ (from the interval $(0, c_R)$ possibly), there will be $\alpha = 0$ and the above behaviour of the functions for the case of the homogeneous medium holds for this value of the velocity even for a piecewise-homogeneous medium /1,4/. When the elastic parameters of medium 2 vary ($c = \mathrm{const}$) the lengths L and l vary between the values $L = \infty$, $l = 0$ (identical media) and the values $l_* = \max l$, $L_* = \min L$ (medium 2 is rigid). Let us compare the solutions for a homogeneous medium and the limit case of a piecewise-homogeneous medium: medium 2 is rigid, $\mu_2, c_{12}, c_{22} \to \infty \Rightarrow p \to b_{11}, q \to b_{21}, d \to a_1$ ($c = \mathrm{const}$). Let a be fixed. Then we obtain for the ratio of the contact pressures that they are half in the case of a homogeneous medium. The fluxes w and the force Q are similarly related. If the energy flux $w(a)$ is fixed in the comparison, then the contact pressures, the fluxes $w(\pm 1)$, and the force Q will be identical, while the crack lengths are different (the crack length is less for a homogeneous medium).

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A SELFSIMILAR PROBLEM ON THE ACTION OF A SUDDEN LOAD ON THE BOUNDARY OF AN ELASTIC HALF-SPACE*

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The solution of the non-linear problem of the action of a constant stress suddenly applied to the plane boundary of an elastic half-space that has homogeneous prestrain is investigated. The problem is selfsimilar, and its solution is constructed from shock and selfsimilar simple waves investigated earlier /1-5/. The problem under consideration is the necessary element that should be contained in solutions of different non-stationary problems, for instance, in the problem of the decay of an arbitrary initial discontinuity. Moreover, the selfsimilar solution constructed below represents the asymptotic form long times of the corresponding non-selfsimilar problems when the stress on the half-space boundary varies from some values to others according to an arbitrary law over a limited time.

1. Formulation of the problem.

A homogeneous isotropic non-linearly elastic medium is given by its internal energy $U(\varepsilon_{ij}, S)$ in the form /1-5/

$$\Phi = \rho_0 U = {}^1/_2\lambda I_1^2 - \mu I_2 - \beta I_1 I_2 + \gamma I_3 - \delta I_1^3 - \xi I_2^2 + \rho_0 T_0 (S - S_0) + \mathrm{const} \tag{1.1}$$

$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_k}{\partial \xi_i}\frac{\partial w_k}{\partial \xi_j}\right)$$

Here S is the entropy, ε_{ij} are the components of Green's strain tensor, w_i is the displacement vector, ρ_0 is the density in the unstressed state, and ξ_i are the Lagrange coordinates that are rectangular Cartesian coordinates in the unstressed state.

The medium that possesses a small homogeneous initial strain occupies the half-space $\xi_3 \geqslant 0$. At the time $t = 0$ a stress that alters the state of strain on the boundary is applied to the boundary $\xi_3 = 0$ and later remains constant. The problem is selfsimilar, and the solution depends on ξ_3, t. A perturbation from the boundary in the domain $\xi_3 > 0$ propagates in the form of plane strain waves in which only the following components of the displacement gradient vary: $\partial w_1/\partial \xi_3 = u$, $\partial w_2/\partial \xi_3 = v$, $\partial w_3/\partial \xi_3 = w$. We designate by U, V, w°, respectively, the initial magnitudes of these strain components, and we denote those values which they acquire on the boundary subjected to the action of the suddenly applied stress by u_*, v_*, w_*, respectively.

In addition to the above, the medium also possesses other strain components that do not vary in this problem and play the part of parameters. These components are ε_{11} and ε_{22}. The

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